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Poly-Essential and General Hyperelastic World (Brane) Models

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Abstract This article provides a unified treatment of an extensive category of non-linear classical field models whereby the universe is represented (perhaps as a brane in a higher dimensional background) in terms of a structure of a mathematically convenient type describable as hyperelastic, for which a complete set of equations of motion is provided just by the energy-momentum conservation law. Particular cases include those of a perfect fluid in quintessential backgrounds of various kinds, as well as models of the elastic solid kind that has been proposed to account for cosmic acceleration. It is shown how an appropriately generalised Hadamard operator can be used to construct a symplectic structure that controles the evolution of small perturbations, and that provides a characteristic equation governing the propagation of weak discontinuities of diverse (extrinsic and extrinsic) kinds. The special case of a poly-essential model—the k-essential analogue of an ordinary polytropic fluid—is examined and shown to be well behaved (like the fluid) only if the pressure to density ratio w is positive.

1 Introduction

As a generalisation of the category of media that are elastic in the ordinary variational sense [1, 2], the extensive category of models referred to here as "hyperelastic" is characterised by an action density L that can be formulated as a non-linear function just of a set of scalar $p+1$ scalar fields $\varphi^0, \varphi^1, \dots, \varphi^p$ and their gradient components $\varphi^0{}_{,a}, \varphi^1{}_{,a}, \dots, \varphi^p{}_{,a}$ with respect to coordinates $\bar{x}^a$ $(a = 0, 1, \dots, p)$ on a $p+1$ dimensional worldsheet with codimension $q \geq 0$ in a background space time endowed with Lorentz signature metric that has components $g_{\mu\nu}$ with respect to coordinates x^μ $(\mu = 0, 1, \dots, p+q)$.

This work is intended for the treatment of scenarios of the usual cosmological type with space dimension $p = 3$, and it is set up (using a background tensor formalism of the kind [3] that was originally developed for the treatment of conducting cosmic strings [5]) in such

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a way as to be applicable not just to models of the traditional kind in which the background spacetime dimension is 4, so that the codimension q vanishes, but also to models of more exotic "brane world" varieties, in which the background is of higher dimension, 5 or more.

In a typical cosmological application of such a model, φ^0 would represent a "quintessence scalar" of the kind commonly invoked [6, 7], to account for the apparent observation of cosmic acceleration while the other scalars $\varphi^1, \varphi^2, \varphi^3$ would be interpretable as comoving (Lagrange type) coordinates of a material medium of the "normal" kind which in the simplest case would be of ordinary perfect fluid type. However instead of being a perfect fluid, the "normal" matter characterised by such comoving coordinates could just as well be an elastic solid of the kind envisaged by Bucher and Spergel [8–10].

In a non-cosmological application for which a model of this hyperelastic kind might be used, the "normal" constituent could be that of a solid neutron star crust, within which a freely flowing superfluid neutron current would be characterised by the scalar field, whose gradient would be the neutron momentum covector, $\mu_a = \varphi^0{}_{,a}$. In that case the scalar would have to be of ignorable type (meaning that the Lagrangian would depend just on its gradient but not on its undifferentiated value φ^0) so that, according to (43), the neutron current would automatically be conserved. (It is however to be remarked that in a realistic treatment of a neutron star crust it may be necessary to allow for the possibility [11] that the superfluid neutron constituent may not be separately conserved, so that a more elaborate kind of model would then be needed.)

Whatever its dimension, the background spacetime metric will induce a corresponding worldsheet metric with components

$$\bar{g}_{ab} = g_{\mu\nu} x^{\mu}{}_{,a} x^{\nu}{}_{,b}, \tag{1}$$

and with determinant $|\bar{g}|$ in terms of which the action integral will be expressible as

$$\mathcal{I} = \int L \|\bar{g}\|^{1/2} \mathrm{d}^{p+1}\bar{x}. \tag{2}$$

In order for the system to be considered as regular, the induced metric on the worldsheet must of course itself have a Lorentz signature, and furthermore the scalar field gradients must be linearly independent so that the fields themselves will be adoptable as an admissible set of worldsheet coordinates, for which we shall simply have $\bar{x}^a = \varphi^a$, which entails that the Lagrangian density L will depend just on the undifferentiated position coordinates φ^a and on the corresponding set of $p(p+1)/2$ induced metric components $\bar{g}_{ab}$.

A subcategory of particular interest consists of models for which the Lagrangian contributions from "scalar" and "normal" parts separate as a sum of the form $L = L_{\mathrm{S}} + L_{\mathrm{N}}$, in which the "normal" part L_{N} is independent of φ^0 and of $\bar{g}_{0a}$, while the "scalar" part is a (non-linear) function only of φ^0 and of its squared gradient as given by the single induced metric component $\bar{g}_{00}$.

2 Dynamics of a Regular Hyperelastic System

Whichever coordinate system is used, the worldsheet stress energy density tensor will have components given [3] by

$$T^{ab} = 2\|\bar{g}\|^{-1/2} \frac{\partial(\|\bar{g}\|^{1/2} L)}{\partial \bar{g}_{ab}}. \tag{3}$$

and there will be a corresponding world hyper-elasticity tensor defined on the worldsheet in the manner introduced by Friedman and Schutz [2] as

$$\mathfrak{C}^{abcd} = \|\bar{g}\|^{-1/2} \frac{\partial(\|\bar{g}\|^{1/2} T^{ab})}{\partial g_{cd}} = \mathfrak{C}^{cdab}, \tag{4}$$

which (modulo a proportionality factor of -2) is interpretable as a relativistic extension of an ordinary (purely spacelike) Cauchy type elasticity E^{abcd} such as will be discussed below.

These worldsheet tensors will map naturally into corresponding background space time tensors given by the expressions

$$T^{\mu\nu} = T^{ab} x^{\mu}{}_{,a} x^{\nu}{}_{,b} = 2\frac{\partial L}{\partial \bar{g}_{\mu\nu}} + L\bar{g}^{\mu\nu}, \tag{5}$$

$$\mathfrak{C}^{\mu\nu\rho\sigma} = \mathfrak{C}^{abcd} x^{\mu}{}_{,a} x^{\nu}{}_{,b} x^{\rho}{}_{,c} x^{\sigma}{}_{,d} = \frac{\partial T^{\mu\nu}}{\partial \bar{g}_{\rho\sigma}} + \frac{1}{2} T^{\mu\nu} \bar{g}^{\rho\sigma}, \tag{6}$$

in which $\bar{g}^{\mu\nu}$ denotes the (first) fundamental tensor of the worldsheet as defined [3] in terms of the contravariant version $\bar{g}^{ab}$ of the induced metric by the formula

$$\bar{g}^{\mu\nu} = \bar{g}^{ab} x^{\mu}{}_{,a} x^{\nu}{}_{,b}. \tag{7}$$

When the codimension q vanishes the overline is redundant here as (7) will then just give back the contravariant version $g^{\mu\nu}$ of the background spacetime metric, and the corresponding mixed version $\bar{g}^{\mu}{}_{\nu}$ will then be just the same as the Kronecker unit tensor δ^{μ}_{ν}, but in a background of higher dimension $n > p + 1$ this mixed version $\bar{g}^{\mu}{}_{\nu}$ will be a non trivial (rank $p + 1$) projector mapping vectors onto their world sheet tangential parts, and giving a corresponding world sheet gradient operator

$$\bar{\nabla}_{\nu} = \bar{g}^{\mu}{}_{\nu} \nabla_{\mu}, \tag{8}$$

in which the distinguishing overline is again redundent in, but only in, the case of vanishing codimension q.

As in more general brane models [3], when the variational field equations ensuring invariance of the action with respect to localised perturbations of the worldsheet and the dynamical fields theron are satisfied, it will automatically follow as a Noether identity that the stress energy given by (3) will satisfy a divergence condition of the standard form

$$\bar{\nabla}_{\mu} T^{\mu}{}_{\nu} = 0. \tag{9}$$

What distinguishes hyperelastic models from others of a more general kind is that in the hyperelastic case no other evolution equations are needed: by itself (9) is not only necessary but will also be sufficent to ensure that the variational field equations are all satisfied.

The way this works is that, to start with, the evolution of the world sheet location (which will only be needed if the codimension q is non-zero) will be governed as always [3] by the orthogonal projection of (9) which will take the standard form

$$T^{\mu\nu} K_{\mu\nu}{}^{\rho} = 0, \tag{10}$$

in which $K_{\mu\nu}{}^{\rho}$ is the second fundamental tensor as defined [3] by

$$K_{\mu\nu}{}^{\rho} = \bar{g}^{\sigma}{}_{\nu} \bar{\nabla}_{\mu} \bar{g}^{\rho}{}_{\sigma}. \tag{11}$$

As well as the Weingarten integrability condition $K_{\mu\nu}{}^{\rho} = K_{\nu\mu}{}^{\rho}$ this defining relation ensures the worldsheet orthogonality condition $K_{\mu\nu}{}^{\rho}\bar{g}^{\sigma}{}_{\rho} = 0$, which evidently entails that $K_{\mu\nu}{}^{\rho}$ itself will vanish in the traditional $q = 0$ case, for which there are no external dimensions so that $\bar{g}^{\sigma}{}_{\rho}$ will just be the identity matrix $\delta^{\sigma}{}_{\rho}$.

When the condition (10) (involving q independent component equations) has been satisfied, the remaining part of (9) will consist just of its tangentially projected part, namely the set of $p+1$ internal component equations given by

$$\bar{g}^{\nu}{}_{\rho}\bar{\nabla}_{\mu}T^{\mu}{}_{\nu} = 0, \tag{12}$$

which is just what is needed to determine the evolution of the $p+1$ independent worldsheet scalars φ^{a}.

3 Symplectic Perturbation Currents and Characteristic Equation

The sufficiency of the divergence condition (9) as a complete set of equations of motion can be understood as a consequence of the feature that with respect to a reference system of the prefered kind in which the scalars ϕ^{a} are used directly as worldsheet coordinates, the configuration of the system will be fully determined just by the specification of the background coordinates x^{ν} as fonctions of these scalars.

It follows that, with respect to such a preferred system, a perturbation of the configuration will be fully determined just by the specification of the corresponding background coordinate displacement, $\delta x^{\mu} = \xi^{\mu}$ say, which, to be dynamically admissible, must of course be such as to satisfy the linear evolution equation obtained as the first order perturbation of (9). When another particular solution, $\delta x^{\mu} = \eta^{\mu}$ say, is already available (for example as a trivial perturbation generated by a symmetry of the system) then the linear equation governing a generic perturbation ξ^{μ} will be conveniently expressible as the conservation,

$$\bar{\nabla}_{\nu}\Omega^{\nu} = 0, \tag{13}$$

of a symplectic worldsheet surface current (such as is also of interest [12] for the purpose of quantisation) of the kind that has recently been shown to be straightforwardly constructable for a widely extended category of brane systems [4]. As in the case of applications [5] to conducting cosmic strings, so also in the more general hyperelastic systems considered here, the possibility of expressing the perturbation just in terms of a displacement vector ξ^{μ} allows the symplectic current Ω^{ν} to be given explicitly by an expression of the form

$$\Omega^{\nu}\{\vec{\xi},\vec{\eta}\} = \eta^{\mu}\mathfrak{O}_{\mu}{}^{\nu}\{\vec{\xi}\} - \xi^{\mu}\mathfrak{O}_{\mu}{}^{\nu}\{\vec{\eta}\}, \tag{14}$$

in terms of what I shall refer to as the hyper-Hadamard operator, which is a linear differential operator whose action on the vector field $\vec{\xi}$ is given by a prescription of the form

$$\mathfrak{O}_{\mu}{}^{\nu}\{\vec{\xi}\} = \mathfrak{H}_{\mu}{}^{\nu}{}_{\rho}{}^{\sigma}\bar{\nabla}_{\sigma}\xi^{\rho} \tag{15}$$

in terms of a corresponding hyper-Hadamard tensor that can be seen [4] to be given in terms of the hyper Cauchy tensor (6) by the formula

$$\mathfrak{H}_{\mu}{}^{\nu}{}_{\rho}{}^{\sigma} = g_{\mu\rho}T^{\nu\sigma} + 2\mathfrak{C}_{\mu}{}^{\nu}{}_{\rho}{}^{\sigma}. \tag{16}$$

It is to be remarked that the standard decomposition

$$g_{\mu\rho} = \bar{g}_{\mu\nu} + \perp_{\mu\nu}, \tag{17}$$

of the background metric into respectively worldsheet tangential and worldsheet orthogonal parts $\bar{g}_{\mu\nu}$ and $\perp_{\mu\nu}$ (of which the latter will vanish when the codimension q is zero) will engender a corresponding decomposition

$$\mathfrak{H}_{\mu\ \rho}^{\ \nu\ \sigma} = \bar{\mathfrak{H}}_{\mu\ \rho}^{\ \nu\ \sigma} + \mathfrak{H}_{\mu\ \rho}^{\perp\nu\ \sigma}, \tag{18}$$

in which the world sheet tangential part is given by

$$\bar{\mathfrak{H}}^{\mu\nu\rho\sigma} = \bar{\mathfrak{H}}^{abcd} x^{\mu}{}_{,a} x^{\nu}{}_{,b} x^{\rho}{}_{,c} x^{\sigma}{}_{,d}, \qquad \bar{\mathfrak{H}}^{abcd} = \bar{g}^{ac} T^{bd} + 2\mathfrak{C}^{abcd}, \tag{19}$$

and the remainder in (18) is given simply by

$$\mathfrak{H}_{\mu\ \rho}^{\perp\nu\ \sigma} = \perp_{\mu\rho} T^{\nu\sigma}. \tag{20}$$

As in the simple elastic case [13], one can obtain the characteristic equation governing the propagation of a discontinuity across a worldsheet hypersurface with normal covector

$$\lambda_a = \lambda_\mu x^{\mu}{}_{,a}, \qquad \lambda_\mu \perp^{\mu}{}_{\nu} = 0, \tag{21}$$

of the second derivative of the perturbation vector $\vec{\xi}$, using the Hadamard rule to the effect that it must be given by an expression the form

$$[\bar{\nabla}_\mu \bar{\nabla}_\nu \xi^\rho] = \lambda_\mu \lambda_\nu \zeta^\rho, \tag{22}$$

in which the vector $\vec{\zeta}$ is a measure (whose calibration depends on the normalisation of the—possibly null—characteristic covector λ_a) of the amplitude of the discontinuity. It can be seen to follow from this rule that the discontinuity of the divergence of the symplectic current (14) will have the form

$$[\bar{\nabla}_\nu \Omega^\nu\{\vec{\xi}, \vec{\eta}\}] = \eta^\mu \mathfrak{H}_{\mu\ \rho}^{\ \nu\ \sigma} \lambda_\nu \lambda_\sigma \zeta^\rho. \tag{23}$$

Since the conservation law (13) is applicable to an arbitrary reference perturbation solution $\vec{\eta}$ (for which undifferentiated components η^μ may be chosen without restriction at any single given point) it can be seen that the characteristic covector λ_μ must satisfy the condition

$$\mathfrak{H}_{\mu\ \rho}^{\ \nu\ \sigma} \lambda_\nu \lambda_\sigma \zeta^\rho = 0. \tag{24}$$

For extrinsic perturbations of the worldsheet location, as obtained by taking $\vec{\zeta}$ to be worldsheet orthogonal, it can be seen from (20) that we simply recover the result (which is already known [3] to apply for a generic, not just hyperelastic, brane model) that the characteristic covector must be a null eigenvector of the stress energy tensor:

$$\bar{g}_{\mu\nu} \zeta^\nu = 0 \quad \Rightarrow \quad T^{\mu\nu} \lambda_\mu \lambda_\nu = 0. \tag{25}$$

On the other hand for tangential perturbations—the only kind that can exist when the codimension q is zero—the characteristic equation will be expressible in purely worldsheet tensorial form:

$$\zeta^\mu = x^{\mu}{}_{,a} \zeta^a \quad \Rightarrow \quad \mathcal{Q}_{ac} \zeta^c = 0, \tag{26}$$

with

$$\mathcal{Q}_{ac} = \mathcal{Q}_{ca} = \bar{\mathfrak{H}}_{a\ c}^{\ b\ d}\lambda_b\lambda_d. \tag{27}$$

The condition for λ_a to be an intrinsic characteristic covector is thus that the determinant $|\mathcal{Q}|$ of this symmetric matrix (27) should vanish.

4 Standard Flow Decomposition

To qualify as hyperelastic in the strictest sense, the system should include a subsystem of ordinary elastic type, as characterised by the requirement that all the scalar fields except one, φ^0 say, should have spacelike gradients and therefore timelike worldsheets that intersect on a congruence of timelike worldlines, whose unit world tangent vector, with components u^a say, will be specified, for $A = 1, \ldots, p$, by

$$u^a\varphi^A{}_{,a} = 0, \qquad \bar{g}_{ab}u^a u^b = -1. \tag{28}$$

Subject to the understanding that the small letters refer to an arbitrary worldsheet coordinate system, but that the capitals refer to a system of the preferred type specified by setting $\bar{x}^0 = \varphi^0$ and $\bar{x}^A = \varphi^A$, the induced metric components on which the action depends can then be listed in a system of the latter type as

$$\bar{g}^{00} = \bar{g}^{ab}\varphi^0{}_{,a}\varphi^0{}_{,b}, \qquad \bar{g}^{A0} = \bar{g}^{ab}\varphi^A{}_{,a}\varphi^0{}_{,b}, \tag{29}$$

and

$$\bar{g}^{AB} = \bar{g}^{ab}\varphi^A{}_{,a}\varphi^B{}_{,a} = \gamma^{ab}\varphi^A{}_{,a}\varphi^B{}_{,a} = \gamma^{AB}, \tag{30}$$

in which the space projected part of the metric is specified in of the usual way as

$$\gamma^{ab} = \bar{g}^{ab} + u^a u^b. \tag{31}$$

The generic action variation will be expressible in terms of these quantities in the form

$$\delta L = \frac{\partial L}{\partial\varphi^0}\delta\varphi^0 + \frac{\partial L}{\partial\varphi^A}\delta\varphi^A + L_{00}\,\delta\bar{g}^{00} + 2L_{A0}\delta\bar{g}^{A0} + L_{AB}\,\delta\bar{g}^{AB}, \tag{32}$$

which provides the components with respect to the preferred system of the worldsheet tensor

$$L_{ab} = \frac{\partial L}{\partial\bar{g}^{ab}} \tag{33}$$

whose contravariant version $L^{ab} = \bar{g}^{ac}\bar{g}^{bd}L_{cd}$ provides the partial derivatives

$$\frac{\partial L}{\partial\bar{g}_{ab}} = -L^{ab} \tag{34}$$

that are needed for the evaluation of the expression (3) for the stress energy tensor, which works out as

$$T^{ab} = L\bar{g}^{ab} - 2L^{ab}. \tag{35}$$

At the next order one obtains the tensor

$$L_{abcd} = \frac{\partial L_{ab}}{\partial \bar{g}^{cd}} = \frac{\partial^2 L}{\partial \bar{g}^{ab} \partial \bar{g}^{cd}} \tag{36}$$

which provides the partial derivatives

$$\frac{\partial^2 L}{\partial \bar{g}_{ab} \partial \bar{g}_{cd}} = L^{abcd} + L^{a(c} \bar{g}^{d)b} + \bar{g}^{a(c} L^{d)b} \tag{37}$$

that are needed for the evaluation of the expression (4) for the hyper-Cauchy tensor, which is thereby obtained as

$$\mathfrak{C}^{abcd} = 2(L^{abcd} + L^{a(c} \bar{g}^{d)b} + \bar{g}^{a(c} L^{d)b}) - (L^{ab} \bar{g}^{cd} + \bar{g}^{ab} L^{cd}) + \frac{L}{2}(\bar{g}^{ab} \bar{g}^{cd} - 2\bar{g}^{a(c} \bar{g}^{d)b}). \tag{38}$$

According to (19) the hyper-Hadamard tensor will therefore be given (using square brackets for index antisymmetrisation) by the expression

$$\bar{\mathfrak{H}}^{abcd} = 4L^{abcd} + 2L^{ac} \bar{g}^{db} + 2L^{a[d} \bar{g}^{b]c} + 2\bar{g}^{a[d} L^{b]c} + L\bar{g}^{a[d} \bar{g}^{b]c}, \tag{39}$$

from which it can be seen that the characteristic matrix (27) will be obtained as

$$\mathcal{Q}_{ac} = 2(L_{ac}\, \bar{g}^{bd} + 2L_{a\ c}^{\ b\ d})\lambda_b \lambda_d. \tag{40}$$

The expansion (32) provides a corresponding Eulerian variation—as carried out for an undisplaced worldsheet location at a fixed value of the background coordinates x^μ and metric $g_{\mu\nu}$—that will be expressible in the form

$$\delta_{\rm E} L = \frac{\partial L}{\partial \varphi^0}\, \delta\varphi^0 + \frac{\partial L}{\partial \varphi^A}\, \delta\varphi^A + J_0^{\ a} \delta\varphi^0{}_{,a} + J_A^{\ a} \delta\varphi^A{}_{,a}, \tag{41}$$

in terms of current vectors given by

$$J_0^{\ a} = 2\bar{g}^{ab}\, (L_{00}\varphi^0{}_{,b} + L_{A0}\varphi^A{}_{,b}), \qquad J_A^{\ a} = 2\bar{g}^{ab}(L_{A0}\varphi^0{}_{,b} + L_{AB}\varphi^B{}_{,b}), \tag{42}$$

which, according to the variational field equations, will have to satisfy worldsheet divergence conditions of the form

$$\bar{\nabla}_a J_0^{\ a} = \frac{\partial L}{\partial \varphi^0}, \qquad \bar{\nabla}_a J_A^{\ a} = \frac{\partial L}{\partial \varphi^A}. \tag{43}$$

5 The Separated Case

In many of the applications of interest a substantial simplification will be provided by the separation of the action density as a sum of the form

$$L = L_{\rm S} + L_{\rm N} \tag{44}$$

in which the "normal" part, $L_{\rm N}$, is independent of φ^0, and $\varphi^0{}_{,a}$ so that it has the form of the action density of an ordinary elastic medium as specified [1, 14] as a function just of φ^A

and $\bar{g}^{AB}$, while the remainder $L_{\rm S}$ is just the action of a simple, albeit non-linear, scalar field model, means that it depends only on φ^0 and on the magnitude, μ say, of its gradient 1-form

$$\mu_a = \varphi^0{}_{,a}, \tag{45}$$

as given by

$$\bar{g}^{00} = \bar{g}^{ab}\mu_a\mu_b = -\mu^2. \tag{46}$$

In such a system, neither part will have any dependence on $\bar{g}^{A0}$, so the cross terms (providing the effect known as "entrainment") in the currents (42) will vanish,

$$L_{A0} = 0. \tag{47}$$

The two subsystems will thus be effectively decoupled, except to the extent that they interact via their effect on the worldsheet location itself whenever the codimension q is non zero, or alternatively, if the codimension is zero, is zero, via their gravitational coupling, which can in that case can be easily incorporated (by requiring the background field $g_{\mu\nu}$ to satisfy Einstein's equations or some generalisation thereof) but which is not so easy to deal with in a higher dimensional background (for which suitable methods of regularisation [15, 16] may be needed).

The other components of the tensor L_{ab} in the separated system will be given by

$$L_{00} = \frac{\partial L_{\rm S}}{\partial \bar{g}^{00}} = -\frac{\partial L_{\rm S}}{\partial(\mu^2)}, \tag{48}$$

and by

$$L_{AB} = \frac{\partial L_{\rm N}}{\partial \bar{g}^{AB}} = \frac{1}{2}(L_{\rm N}\gamma_{AB} - P_{AB}), \tag{49}$$

where γ_{AB} is the covariant inverse of the contravariant base space metric $\gamma^{AB} = \bar{g}^{AB}$ defined by (30), and P_{AB} will be interpretable as the correspondingly index lowered version of the pressure tensor P^{AB} of the medium, from which its elasticity tensor will be obtainable in the usual way [1] as

$$E^{AB}{}_{CD} = 2\frac{\partial P^{AB}}{\partial \bar{g}^{CD}} - P^{AB}\gamma_{CD}. \tag{50}$$

This means that with respect to arbitrary worldsheet coordinates the pressure tensor $P_{ab} = P_{AB}\varphi^A{}_{,a}\varphi^B{}_{,b}$ of the elastic medium will have a contravariant version expressible as

$$P^{ab} = L_N\gamma^{ab} - 2\gamma^{ac}\gamma^{bd}L_{cd} \tag{51}$$

while the ordinary elasticity tensor of the medium will be given by

$$E^{abcd} = L_{\rm med}(\gamma^{ab}\gamma^{bc} - 2\gamma^{a(c}\gamma^{d)b}) + P^{a(c}\gamma^{d)b} + \gamma^{a(c}P^{d)b} - 4\gamma^{ae}\gamma^{bf}\gamma^{cg}\gamma^{dh}L_{efgh}. \tag{52}$$

It follows from (35) that the total stress energy tensor will be given by the sum

$$T^{ab} = T^{ab}_{\rm S} + T^{ab}_{\rm N}, \tag{53}$$

in which the scalar field contribution will be given by

$$T_{\rm S}^{ab} = L_{\rm S}\,\bar g^{ab} - 2L_{\rm S}^{ab}, \quad L_{\rm S}^{ab} = -L'_{\rm S}\mu^a\mu^b,\ L'_{\rm S} = \frac{\partial L_{\rm S}}{\partial(\mu^2)}, \tag{54}$$

which means that the scalar field will have a pressure $P_{\rm S}$ and rest frame energy density $\rho_{\rm S}$ given by

$$P_{\rm S} = L_{\rm S}, \qquad \rho_{\rm S} = 2\mu^2 L'_{\rm S} - L_{\rm S}, \tag{55}$$

while the contribution from the "normal" part will be given by an expression of the standard form

$$T_{\rm N} = \rho_{\rm N}\,u^a u^b + P^{ab}, \quad \rho_{\rm N} = -L_{\rm N}. \tag{56}$$

6 Separated Characteristic Equations

Like the stress tensor, so also the hyperelasticity tensor will be expressible in the separated case as a sum

$$\mathfrak{C}^{abcd} = \mathfrak{C}_{\rm S}^{abcd} + \mathfrak{C}_{\rm N}^{abcd}, \tag{57}$$

in which, according to (38), the scalar field contribution will be given by

$$\mathfrak{C}_{\rm S}^{abcd} = 2(L_{\rm S}^{abcd} + L_{\rm S}^{a(c}\bar g^{d)b} + \bar g^{a(c}L^{d)b}) - (L_{\rm S}^{ab}\bar g^{cd} + \bar g^{ab}L_{\rm sca}^{cd}) + \frac{L_{\rm S}}{2}(\bar g^{ab}\bar g^{cd} - 2\bar g^{a(c}\bar g^{d)b}), \tag{58}$$

with

$$L_{\rm S}^{abcd} = L''_{\rm S}\mu^a\mu^b\mu^c\mu^d, \qquad L''_{\rm S} = \frac{\partial L'_{\rm S}}{\partial(\mu^2)}, \tag{59}$$

while for the contribution from the "normal" elastic medium it can be seen that we shall recover the Friedman Schutz [2] formula

$$\mathfrak{C}_{\rm N}^{abcd} = -\frac{1}{2}E^{abcd} + P^{a(c}u^{d)}u^b + u^a u^{(c}P^{d)b} - \frac{1}{2}(P^{ab}u^c u^d + u^a u^b P^{cd}) + \frac{1}{2}\rho(\bar g^{ab}\bar g^{cd} - 2\bar g^{a(c}\bar g^{d)b}). \tag{60}$$

In the separated case the crossed element, with respect to the preferred coordinate system, of the characteristic matrix (27) will automatically vanish

$$\mathcal{Q}_{0A} = 0, \tag{61}$$

and there will be a decoupling of the "normal" characteristic modes, for which the discontinuity amplitude ζ^a is orthogonal to μ_a, from the "scalar" characteristic mode for which ζ^a is aligned with u^a. The latter is given by

$$\zeta^A = 0 \quad\Rightarrow\quad \mathcal{Q}_{00} = 0, \tag{62}$$

so that one obtains a scalar characteristic equation in the form

$$(2L''_{\rm S}(\mu^a\lambda_a)^2 - L'_{\rm S}\lambda^2, \lambda^2 = \bar g^{ab})\lambda_a\lambda_b = 0. \tag{63}$$

As the relative propagation speed $v_{\rm S}$ will be definable using a standard calibration of the characteristic covector λ_a by

$$v_{\rm S}^2 = (\mu^a\lambda_a)^2/\mu^2, \qquad \lambda^2 = 1 - v_{\rm S}^2, \tag{64}$$

it can be seen that the characteristic equation just gives the condition

$$v_{\rm S}^{-2} = 1 + 2\mu^2 L''_{\rm S}/L'_{\rm S}. \tag{65}$$

It can thus be seen that for a scalar model of the "standard" kind with a linear equation of state, meaning one with $L''_{\rm S} = 0$, the discontinuities will travel at the speed of light, $v_{\rm S}^2 = 1$, while for other models of k-essence [17] or more general kinds [18] it can be seen that the condition for causality, $v_{\rm S}^2 \le 1$, and the reality condition for local stability, $v_{\rm S}^2 \ge 0$, will both be satisfied if and only if the scalar equation of state is such that

$$2\mu^2 L''_{\rm S}/L'_{\rm S} \ge 0. \tag{66}$$

As well as these simple "scalar" characteristic modes (not to mention the extrinsic characteristic modes that may exist, subject to (25), if there is a higher dimensional background) there will be different kinds of "normal" characteristic modes given by

$$\zeta^0 = 0 \quad \Rightarrow \quad \mathcal{Q}_{AC}\zeta^C = 0, \tag{67}$$

so that using the standard calibration

$$\lambda_a = \nu_a + v u_a, \qquad \nu_a u^a = 0, \qquad \nu^a\nu_a = 1, \tag{68}$$

to define the relative propagation velocity v and propagation direction ν^a (modulo a sign that may be chosen to make v positive) the corresponding characteristic condition on the normal covector λ_a will be the vanishing of the p-dimensional determinant of the matrix whose components can be seen from (40) to be given by

$$\mathcal{Q}_{AC} = 2(L_{AC}\lambda^2 + 2L_{ABCD}\nu^B\nu^D), \quad \lambda^2 = 1 - v^2. \tag{69}$$

For the simple fluid case—as given by the restriction that $L_{\rm N}$ should depend only on the components of the base metric γ_{AB} only via its determinant $|\gamma|$—one will obtain

$$L_{AB} = -L^{\wr}_{\rm N}\gamma_{AB}, \quad L^{\wr}_{\rm N} = \frac{\partial L_{\rm N}}{\partial(\ln|\gamma|)} = |\gamma|\frac{\partial L_{\rm N}}{\partial|\gamma|}, \tag{70}$$

so the pressure tensor will have the isotropic form,

$$P^{ab} = P_{\rm N}\gamma^{ab}, \tag{71}$$

in which the ordinary pressure scalar will be given by the well known (but in my previous review [14] miscopied) formula

$$P_{\rm N} = L_{\rm N} + 2L^{\wr}_{\rm N}. \tag{72}$$

In terms of the corresponding bulk modulus, namely

$$\beta_{\rm N} = -2P^{\wr}_{\rm N} = -2|\gamma|\frac{\partial P_{\rm N}}{\partial|\gamma|} = -2L^{\wr}_{\rm N} - 4L^{\wr\wr}_{\rm N}, \tag{73}$$

one will obtain

$$L_{ABCD} = -\frac{1}{4}\beta\gamma_{AB}\gamma_{CD} + \frac{1}{2}L^{\wr}_{\rm N}\gamma_{AC}\gamma_{BD} + L^{\wr}_{\rm N}\gamma_{A[D}\gamma_{B]C}, \tag{74}$$

and the elasticity tensor will be given [14] by

$$E^{abcd} = (\beta_{\rm N} - P_{\rm N})\gamma^{ab}\gamma^{cd} + 2P_{\rm N}\gamma^{a(c}\gamma^{d)b}. \tag{75}$$

It can thus be seen that—as in the more general case of an isotropic solid configuration [10]—the characteristic equation (67) for the "normal" fluid will have solutions of two kinds, namely longitudinal modes with $v = v_{\rm L}$, and transverse modes with $v = v_{\rm S}$, of which the latter are non propagating in the fluid—as opposed to solid—case,

$$\zeta^A\nu_A = 0 \quad\Rightarrow\quad v^2 = v_{\rm S}^2, \quad v_{\rm S}^2 = 0, \tag{76}$$

while the former are just ordinary sound waves,

$$\zeta^A = \nu^A \quad\Rightarrow\quad v^2 = v_{\rm L}^2, \quad v_{\rm L}^2 = \beta_{\rm N}/(\rho_{\rm N} + P_{\rm N}). \tag{77}$$

7 Discussion: Polytropic and Poly-Essential Equations of State

For both a "normal" fluid and a "scalar" constituent, an important role is played by their respective pressure to density ratios, namely

$$w_{\rm N} = P_{\rm N}/\rho_{\rm N}, \qquad w_{\rm S} = P_{\rm S}/\rho_{\rm S}, \tag{78}$$

which can be seen to be given by

$$w_{\rm N} = 1 + 2L^{\wr}_{\rm N}/L_{\rm N}, \qquad w_{\rm S}^{-1} = 2\mu^2 L'_{\rm S}/L_{\rm S} - 1. \tag{79}$$

In recent years it has become common, in specialised cosmological (though not general astrophysical) literature, to use a rather loose terminology whereby the pressure to density ratio is referred to as the "equation of state", an appelation that is jusifiable only if the ratio in question is actually constant. If—as will often but not always be a good approximation—it can be supposed that this ratio, $w_{\rm N}$ or $w_{\rm S}$ as the case may be, is constant then it will indeed characterise a corresponding equation of state. In the "normal" fluid case, such an equation of state will be of what is known (in the astrophysical literature) as polytropic type, with polytropic index $\gamma_{\rm N}$, as given an ansatz of the form

$$L_{\rm N} = -C|\gamma|^{-\gamma_{\rm N}/2}, \quad \gamma_{\rm N} = 1 + w_{\rm N}, \tag{80}$$

for some constant coefficient C. In the "scalar" case, the postulate that $w_{\rm S}$ should be constant can be seen to imply that the equation of state will be of what may be termed poly-essential type, as given by a power law ansatz, with index $\alpha_{\rm S}$, of the form

$$L_{\rm S} = \mathfrak{K}\mu^{\alpha_{\rm S}}, \quad \alpha_{\rm S} = 1 + w_{\rm S}^{-1}, \tag{81}$$

in which the coefficient $\mathfrak{K}$ is independent of the field gradient magnitude μ, but is given as some function just of the scalar field magnitude φ^0. This means that the poly-esssential ansatz (81) characterises a special subcategory within the category of quintessential models called k-essential [17]. This subcategory includes a model of the "standard" type, namely the trivial case of a free massless scalar field, in the limit when $w_S = 1$, which corresponds to $\alpha_S = 2$.

Attention in cosmology has centered during recent years on the observational evidence to the effect that the universe is accelerating, which suggests the need [19] for a model with a rather strongly negative value, somewhere in the range between $-1/3$ and -1, for the mean pressure to density ratio w. This poses a problem for a "normal" fluid model of the kind specified for a given value of w_N by the equation of state (80) for which it is well known that the longitudinal modes (79) will have propagation velocity given by

$$v_L^2 = w_N, \tag{82}$$

so that the conditions of reality (for local stability) and causality imply the restrictions

$$0 \le w_N \le 1. \tag{83}$$

It has been rather unfairly suggested that, compared with such ordinary fluid models, k-essential and other scalar models are advantaged by the absence of such a restriction, since they have squared characteristic velocity, v_S^2—as given by (7) of the article [17] referred to—that "can be positive for any" value of the relevant "equation of state" ratio, w_S.

The reason why this is unfair is that if w_S is a bona fide "equation of state" parameter, meaning that it is actually constant, then the corresponding equation of state, namely the poly-essential ansatz (81) entails, according to (65), that the relevant propagation velocity will be given by

$$v_S^2 = (\alpha_S - 1)^{-1} = w_S, \tag{84}$$

which means that the situation will just the same as in the ordinary fluid case, in so much as the corresponding restriction

$$0 \le w_S \le 1. \tag{85}$$

will have to be satisfied.

The way suggested by Bucher and Spergel [8] for getting over the restriction (83) is to a seek a mechanism providing elastic rigidity. As an alternative, the way the advocates of k-essential (and other quintessential) cosmological models [18] propose to get around the restriction (85) is of course to drop the postulate that w_S should be a genuine "equation of state" parameter, and instead use equations of state of more general kinds in which w_S is demoted from the status of a fixed parameter to that of a variable field. What is unfair is to give the impression that the possibity of doing that is a privilege distinguishing scalar field models from "normal" fluid models: in fact the use of more general kinds of ordinary fluid model—in which the density to pressure ratio w_N is just a variable field that may be quite different from the squared sound speed—is actually commonplace in many areas of astrophysics, and should not be prematurely ruled out of consideration in a cosmological context.

8 Conclusions

The category of models presented in the preceeding work is sufficient for a wide range of cosmological applications of the traditional $3 + 1$ kind in which the codimension q is taken to be zero, so that the spacetime metric can be taken to governed by the ordinary Einstein equations.

The treatment here has been set up in such a way as to be applicable also to scenarios in which our 4-dimensional spacetime manifold is considered to be a thin worldbrane in a higher—meaning $4+q$—dimensional background with a given geometry. However for that kind of application it is difficult—despite the efforts of many workers in recent years—to see how to allow realistically for the effect of gravity except in a Cowling type approximation in which the scale of the perturbations is supposed to be sufficiently small (compared with the relevant Jeans length) for their self interactions to be neglected. A much studied (albeit only marginally plausible) possibility—that is beyond the scope of the present treatment but that does include allowance for strong gravitational coupling—is that of the Randall–Sundrum $q=1$ type scenarios [20, 21] (and their reflection symmetry violating generalisations [22, 23]) but the unsatisfactory features of such scenarios include the loss of effective predictability due to incoming gravitational waves from the bulk.

In scenarios with codimension $q \geq 2$ it is even harder to see how to allow properly for gravitation. Attempts to account for the appearance of 4-dimensional gravity as a simulated effect [24] due to acceleration with respect to a fixed bulk geometry tend to predict comportment of scalar—tensor type rather than the pure spin—2 gravity that is actually observed. Progress has however been achieved [15, 16] in the regularisation of the divergences that will result from true gravity in the bulk, and that can be allowed for (if not too strong) within a treatment of the kind presented here as an extra contribution to the effective stress energy tensor appearing in the extrinsic characteristic equation (25). The condition that the extrinsic propagation velocities should always be real and compatible with causality will evidently restrict the admissible values of the eigenvalues of the net stress-energy tensor in (25). In particular, if it is of the isotropic form

$$T^{\mu\nu} = \rho((1+w)u^\mu u^\nu + w\bar{g}^{\mu\nu}) \tag{86}$$

then the extrinsic propagation velocity $v_{\rm E}$ will be given by

$$v_{\rm E}^2 = -w. \tag{87}$$

It follows (unless the codimension q is zero) that instead of being subject to a positivity condition of the familiar kind exemplified by (83) and (85) the net pressure to density ratio w will have to satisfy the negativity condition

$$-1 \leq w \leq 0. \tag{88}$$

Such pressure negativity is not incompatible with the observational evidence, but does exclude the possibility of describing the universe just in terms of scalar fields and fluids that are purely of respectively poly-essential and polytropic type.

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References

1. Carter, B.: Elastic perturbation theory in general relativity and a variational principle for a rotating solid star. Commun. Math. Phys. **30**, 261–286 (1973)

2. Friedman, J.L., Schutz, B.F.: On the stability of relativistic systems. Astrophys. J. **200**, 204–220 (1975)
3. Carter, B.: Essentials of classical brane dynamics. Int. J. Theor. Phys. **40**, 2099–2130 (2001), gr-qc/0012036
4. Carter, B.: Symplectic structure in brane mechanics. Int. J. Theor. Phys. **42**, 1317–1327 (2003), hep-th/0302084
5. Carter, B., Steer, D.A.: Symplectic structure for elastic and chiral conducting cosmic string models. Phys. Rev. D **69**, 125002 (2004), hep-th/0307161
6. DeDeo, S., Caldwell, R.R., Steinhardt, P.J.: Effects of the sound speed of quintessence on the microwave background and large scale structure. Phys. Rev. D **67**, 103509 (2003)
7. DeDeo, S., Caldwell, R.R., Steinhardt, P.J.: Phys. Rev. D **69** 129902 (2004), astro-ph/0301284
8. Bucher, M., Spergel, D.N.: Is the dark matter a solid. Phys. Rev. D **60**, 043505 (1999), astro-ph/9812022
9. Carter, B.: Frozen rigging model of the energy dominated universe. Int. J. Theor. Phys. **44**, 1729–1741 (2005), hep-ph/0412397
10. Battye, R.A., Carter, B., Chachoua, E., Moss, A.: Rigidity and stability of cold dark solid universe model. Phys. Rev. D **72**, 023503 (2005), hep-th/0501244
11. Langlois, D., Sedrakian, D., Carter, B.: Differential rotation of relativistic superfluid in neutron stars. Mon. Not. Roy. Astr. Soc. **297** 1198–1201 (1998), astro-ph/9711042
12. Cartas-Fuentevilla, R.: Identically closed two-form for covariant phase space quantisation of Dirac–Nambu–Goto p-branes in a curved space-time. Phys. Lett. B **536** 283–288 (2002), hep-th/0204133
13. Carter, B.: The speed of sound in a high pressure general relativistic solid. Phys. Rev. D **7**, 1590–1593 (1973)
14. Carter, B.: Interaction of gravitational waves with an elastic solid medium In: Deruelle, N., Piran, T. (eds.) Gravitational Radiation, Les Houches, 1982, pp. 455–464. North-Holland, Amsterdam (1983), gr-qc/0102113
15. Carter, B., Battye, R.A., Uzan, J.P.: Gradient formula for linearly self-interacting branes. Commun. Math. Phys. **235**, 289–311 (2003), hep-th/0204042
16. Battye, R., Carter, B., Mennim, A.: Linearized self-forces for branes. Phys. Rev. D **71**, 104026 (2005), hep-th/0412053
17. Armendariz-Picon, C., Mukhanov, V., Steinhardt, P.J.: Essentials of k-essence. Phys. Rev. D **63**, 103510 (2001), astro-ph/0006373
18. Erickson, J.K., Caldwell, R.R., Steinhardt, P.J., Armendariz-Picon, C., Mukhanov, V.: Measuring the speed of sound of quintessence. Phys. Rev. Lett. **88**, 121301 (2002), astro-ph/0112438
19. Conversi, L., Melchiorri, A., Mersini, L., Silk, J.: Are domain walls ruled out? Astropart. Phys. **21**, 443–449 (2004), astro-ph/0402529
20. Binetruy, P., Deffayet, C., Langlois, D.: Nonconventional cosmology from a brane universe. Nucl. Phys. B **565**, 269–287 (2000), hep-th/9905012
21. Sasaki, M., Shiromizu, T., Maeda, K.: Gravity, stability and energy conservation on the Randall–Sundrum brane world. Phys. Rev. D **62**, 024008 (2000), hep-th/9912233
22. Battye, R.A., Carter, B., Mennim, A., Uzan, J.P.: Einstein equations for an asymmetric brane-world. Phys. Rev. D **64**, 124007 (2001), hep-th/0105091
23. Kunze, K.E., Vazquez-Mozo, M.A.: Quintessential brane cosmology. Phys. Rev. D **65**, 044002 (2002), hep-th/0109038
24. Carter, B., Uzan, J.P., Battye, R.A., Mennim, A.: Simulated gravity without true gravity in asymmetric brane-world scenarios. Class. Quantum Gravity **18**, 4871–4896 (2001), gr-qc/0106038